Math 31B, Lecture 2
Integration and Infinite Series

Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books, notes, or calculators. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

Consider the function $f(x) = x^{\ln x}$.

(a) [5pts.] Find the derivative $f'(x)$.

> **Solution:** We take the natural logarithm of both sides, leading to $\ln(f(x)) = \ln(x^{\ln x}) = \ln(x)^2$. Differentiating on both sides gives $\frac{f'(x)}{f(x)} = \frac{2\ln x}{x}$. We conclude that $f'(x) = \frac{2x^{\ln(x)}\ln x}{x}$.

(b) [5pts.] Let $g(x) = f^{-1}(x)$ be the inverse of f. Find $g'(e)$.

> **Solution:** First (as pointed out by Peize Liu) this question should have said "for $x \geq 1$ to make $f(x)$ one-to-one, hence invertible. With that restriction, observe that $f(e) = e$, so $g(e) = e$. Ergo $g'(e) = \frac{1}{f'(g(e))} = \frac{1}{f'(e)}$. Since $f'(e) = \frac{2 \cdot e^{\ln e} \cdot \ln e}{e} = \frac{2e}{e} = 2$, we conclude that $g'(e) = \frac{1}{2}$.

Problem 2.

We have a sample of 200 grams of a radioactive isotope. After one year it decays to 150 grams.

(a) [5pts.] What is the half-life of the isotope?

> **Solution:** We know that the amount of isotope remaining is given by $P(t) = 200e^{-kt}$, where t is time elapsed in years and k is the decay constant. Since $P(1) = 150$, we see that $150 = 200e^{-k}$, so $\ln(.75) = k$ is the decay constant. Then the half-life, by the formula proved in class, is $\frac{\ln 2}{k} = -\frac{\ln 2}{\ln .75}$.

(b) [5pts.] After how many years will the sample decay to 70 grams?

> **Solution:** We want to find t such that $70 = 200e^{-\ln(.75)t}$. Solving the equation, we see that $t = \frac{\ln(.35)}{\ln(.75)}$.

Problem 3.

The temperature $P(t)$ in degrees Celsius of a hot cup of coffee in a room of constant temperature after t minutes of cooling satisfies the differential equation

$$P'(t) + .05P(t) = 1$$

(a) [5pts.] Find the temperature of the room and the cooling constant k. [Hint: First rewrite the equation in a familiar form.]

Solution: We can rewrite the equation as $P'(t) = .05(20 - P(t))$. This means the temperature of the room is 20 degrees Celsius and the cooling constant is $k = .05$.

(b) [5pts.] Assuming the coffee was served at 80 degrees Celsius, at what time will it be 60 degrees?

Solution: With this initial condition, the solution to the differential equation in part (a) is $P(t) = 20+60e^{.05t}$. Ergo we wish to find t such that $65 = 20+60e^{-.05t}$. Solving the equation, we see that $t = \frac{\ln(\frac{2}{3})}{-.05} = -20\ln(\frac{2}{3})$.

Problem 4.

Calculate the following limits:

(a) [5pts.]

$$\lim_{x\to 0} \frac{x^2}{1-\cos x}$$

Solution: This limit is in the form $\frac{0}{0}$, and the derivative of the denominator is nonzero near (although not at) 0, so we apply L'Hopital's Rule.

$$\lim_{x\to 0} \frac{x^2}{1-\cos x} = \lim_{x\to 0} \frac{2x}{\sin x}$$

The limit on the right is still of the form $\frac{0}{0}$; we apply L'Hopital's Rule a second time.

$$\begin{aligned} \lim_{x\to 0} \frac{2x}{\sin x} &= \lim_{x\to 0} \frac{2}{\cos x} \\ &= \frac{2}{1} \\ &= 2 \end{aligned}$$

(b) [5pts.]

$$\lim_{x\to \frac{\pi}{2}} (x - \frac{\pi}{2}) \tan x$$

Solution: The limit is in the form $0 \cdot \infty$; we change it into a quotient $\lim_{x\to\frac{\pi}{2}} \frac{(x-\frac{\pi}{2})}{\cot x}$. Now this limit is in the form $\frac{0}{0}$, and the derivative of the denominator is nonzero near $\frac{\pi}{2}$, so we may apply L'Hopital's Rule.

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}} \frac{(x-\frac{\pi}{2})}{\cot x} &= \lim_{x\to\frac{\pi}{2}} \frac{1}{-\csc^2(x)}\\ &= \lim_{x\to\frac{\pi}{2}} -sin^2(x)\\ &= -1\end{aligned}$$

Problem 5. *10pts.*
Calculate the definite integral:

$$\int_0^{\sqrt{\frac{\pi}{2}}} x^3\sin(x^2)dx$$

Solution: First, we use substitution. Let $t = x^2$. Then $dt = 2xdx$, and the integral reduces to

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} t\sin(t)dt$$

Now we may proceed by integration by parts. Let $u(t) = t$ and $v'(t) = \sin(t)$. Then $u'(t) = 1$, while $v(t) = -\cos t$, and by integration by parts formula, we have the following.

$$\begin{aligned}\frac{1}{2}\int_0^{\frac{\pi}{2}} t\sin(t)dt &= \frac{1}{2}\left[-t\cos t\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}}\cos tdt\\ &= \frac{1}{2}\left[-\frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)-0\right] - \frac{1}{2}\left[\cos t\right]_0^{\frac{\pi}{2}}\\ &= 0 + [\frac{1}{2}-0]\\ &= \frac{1}{2}\end{aligned}$$